

Some unsolved problems on cycles *

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Abstract

Hajos' conjecture that every simple even graph on n vertices can be decomposed into at most $(n - 1)/2$ cycles (see L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236). Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. P. Erdos raised the problem of determining $f(n)$ (see J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976), p.247, Problem 11). Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turan number. P. Erdos conjectured that there exists a positive constant c such that $ex(n, C_{2k}) \geq cn^{1+1/k}$ (see P. Erdos, Some unsolved problems in graph theory and combinatorial analysis, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), pp. 97–109, Academic Press, London, 1971). This paper summarizes some results on these problems and the conjectures that relate to these. It seems to us that Hajós conjecture is false.

Key words: Hajós conjecture; even graph; Turan number; cycle; the maximum number of edges

AMS Subject Classifications: 05C35, 05C38

1 Hajós conjecture

An *eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. Let G be an eulerian graph. A *circuit decomposition* of G is a set of edge-disjoint circuits C_1, C_2, \dots, C_t such that $E(G) = C_1 \cup C_2 \cup \dots \cup C_t$. It is well known that every eulerian graph has a circuit decomposition. A natural question is to find the smallest number t such that G has a circuit decomposition of t circuits? Such smallest number t is called the *circuit decomposition* number of G , denoted by $cd(G)$. For each edge $xy \in E(G)$, let $m(xy)$ be the number of edges between x and y . The *multiple number* of G is defined by $m(G) = \sum_{uv \in E(G)} (m(uv) - 1)$ (see [12]).

The following conjecture is due to Hajós (see [29]).

Hajós conjecture:

$$cd(G) \leq \frac{|V(G)| - 1}{2}$$

for every simple eulerian graph G .

Lovasz [29] proved that

Theorem 1.1 (Lovasz [29]) A graph of n vertices can be covered by $\leq \lceil n/2 \rceil$ disjoint paths and circuits.

Jiang [19] and Seyffarth [30] proved that

Theorem 1.2 (Jiang [19] and Seyffarth [30]) $cd(G) \leq \frac{|V(G)|-1}{2}$ for every simple planar eulerian graph G .

Granville and Moisiadis [17] and Favaron and Kouider [13] proved that **Theorem 1.3 (Granville and Moisiadis [17] and Favaron and Kouider [13])** If G is an even multigraph of order n , of size m , with $\Delta(G) \leq 4$, then $cd(G) \leq \frac{n+M-1}{2}$ where $M = m - m^*$ and m^* is the size of the simple graph induced by \bar{G} .

Fan and Xu [12] proved that

Theorem 1.4 (Fan and Xu [12]) If G is an eulerian graph with

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

then G has a reduction H such that

$$cd(H) > \frac{|V(H)| + m(H) - 1}{2}$$

and the number of vertices of degree less than six in H plus $m(H)$ is at most one.

Corollary 1.5 (Fan and Xu [12]) Hajós conjecture is valid for projective graphs.

Corollary 1.6 (Fan and Xu [12]) Hajós conjecture is valid for K_6^- minor free graphs.

Xu[40] also proved the following two results:

Theorem 1.7 (Xu[40]) If G is an eulerian graph with

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

such that

$$cd(H) \leq \frac{|V(H)| + m(H) - 1}{2}$$

for each proper reduction of G , then G is 3-connected. Moreover, if $S = \{x, y, z\}$ is a 3-cut of G , letting G_1 and G_2 be the two induced subgraph of G such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$, then either S is not an independent set, or G_1 and G_2 are both eulerian graphs.

Corollary 1.8 (Xu[40]) To prove Hajós' conjecture, it suffices to prove

$$cd(G) \leq \frac{|V(G)| + m(G) - 1}{2}$$

for every 3-connected eulerian graph G .

Fan [11] proved that

Theorem 1.9 (Fan [11]) Every eulerian graph on n vertices can be covered by at most $\lfloor \frac{n-1}{2} \rfloor$ circuits such that each edge is covered an odd number of times.

Xu and Wang[41] give

Theorem 1.10 (Xu and Wang[41]) *The edge set of each even toroidal graph can be decomposed into at most $(n+3)/2$ circuits in $O(mn)$ time, where a toroidal graph is a graph embedable on the torus.*

Theorem 1.11 (Xu and Wang[41]) *The edge set of each toroidal graph can be decomposed into at most $3(n-1)/2$ circuits and edges in $O(mn)$ time.*

It seems to us that Hajós conjecture is false.

2 Erdos Problem

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi[31] proved that

Theorem 2.1 (Shi[31])

$$f(n) \geq n + [(\sqrt{8n-23} + 1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson, and Shreve [4], Jia[18], Lai[21,22,23,24,25,26,27], Shi[32,33,34,35,36,37,38] obtained some results.

Boros, Caro, Füredi and Yuster[3] proved that

Theorem 2.2 (Boros, Caro, Füredi and Yuster[3]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [28] proved that

Theorem 2.3 (Lai [28])

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1 - o(1))$$

and proposed the following conjecture:

Conjecture 2.4 (Lai [28])

$$\lim_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

It seems difficult to prove this conjecture. It would be nice to prove the following weakening conjectures:

Conjecture 2.5 (Lai[23])

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

or

Conjecture 2.6 (Lai[24])

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{2.4}.$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

Shi[34] proved that

Theorem 2.7 (Shi[34]) For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n - 15} - 3)]$.

Chen, Lehel, Jacobson, and Shreve [4] proved that

Theorem 2.8 (Chen, Lehel, Jacobson, and Shreve [4]) $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

Boros, Caro, Füredi and Yuster [3] improved this lower bound significantly:

Theorem 2.9 (Boros, Caro, Füredi and Yuster [3]) $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

Corollary 2.10 (Boros, Caro, Füredi and Yuster [3])

$$\sqrt{2} \geq \limsup \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf \frac{f_2(n) - n}{\sqrt{n}} \geq 1$$

Boros, Caro, Füredi and Yuster [3] made the following conjecture:

Conjecture 2.11 (Boros, Caro, Füredi and Yuster [3])

$$\lim \frac{f_2(n) - n}{\sqrt{n}} = 1.$$

It is easy to see that Conjecture 2.11 implies the (difficult) upper bound in the Erdos Turan Theorem [7,10](see Boros, Caro, Füredi and Yuster [3]).

Markström [20] raised the following problem:

Problem 2.12 (Markström [20]) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ denote the least number of edges of a graph which contains a cycle of length k for every $1 \leq k \leq n$. Jia[18] proved the following results:

Theorem 2.13 (Jia[18])

When n is large,

$$n + \log_2 n - 1 \leq g(n) \leq n + \frac{3}{2} \log_2 n + 1.$$

Theorem 2.14 (Jia[18])

For a large positive integer n , $g(n) \leq n + \log_2 n + \frac{3}{2} \log_2 \log_2 n + O(1)$

Corollary 2.15 (Jia[18])

For n large, $g(n) = n + \log_2 n + O(\log_2 \log_2 n)$.

Jia[18] made the following conjecture:

Conjecture 2.16 (Jia[18])

$$g(n) = n + \log_2 n + O(1),$$

as $n \rightarrow \infty$.

The sequence (c_1, c_2, \dots, c_n) is the cycle length distribution of a graph G of order n where c_i is the number of cycles of length i in G . Let $f(a_1, a_2, \dots, a_n)$ denote the maximum possible number of edges which satisfies $c_i \leq a_i$ where a_i is a nonnegative integer. Shi posed the problem of determining $f(a_1, a_2, \dots, a_n)$ which extended the problem due to Erdos, it is clearly that $f(n) = f(1, 1, \dots, 1)$ (see Xu and Shi[42]).

The lower bound $f(0, 0, 2, \dots, 2)$ is given by Xu and Shi[42].

Theorem 2.17 (Xu and Shi[42])

For $n \geq 3$,

$$f(0, 0, 2, \dots, 2) \geq n - 1 + [(\sqrt{11n - 20})/2],$$

and the equality holds when $3 \leq n \leq 10$.

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turan number.

We denote by $m_i(n)$ the most probable numbers of cycles of length i in the complete graph K_n on n vertices. Obviously,

$$\begin{aligned} ex(n, C_k) &= f(0, 0, m_3(n), \dots, \\ &\quad m_{k-1}(n), 0, m_{k+1}(n), \dots, m_n(n)) \\ &= f(0, 0, 2^{\frac{n(n-1)}{2}}, \dots, \\ &\quad 2^{\frac{n(n-1)}{2}}, 0, 2^{\frac{n(n-1)}{2}}, \dots, 2^{\frac{n(n-1)}{2}}). \end{aligned}$$

Therefore, finding $ex(n, C_k)$ is a special case of determining $f(a_1, a_2, \dots, a_n)$.

3 Erdos conjecture.

P. Erdos conjectured that there exists a positive constant c such that $ex(n, C_{2k}) \geq cn^{1+1/k}$ (see Erdos[9]). Erdos [6] posed the problem of determining $ex(n, C_4)$.

Erdos [8] and Bondy and Simonovits [2] obtained that

Theorem 3.1 (Erdos [8] and Bondy and Simonovits [2])

$$ex(n, C_{2k}) \leq cn^{1+1/k}$$

Wenger [39] proved that

Theorem 3.2 (Wenger [39])

$$ex(n, C_4) \geq \left(\frac{n}{2}\right)^{3/2},$$

$$ex(n, C_6) \geq \left(\frac{n}{2}\right)^{4/3},$$

$$ex(n, C_{10}) \geq \left(\frac{n}{2}\right)^{6/5}$$

Furedi[14] proved that

Theorem 3.3 (Furedi[14]) If q is a power of 2, then

$$ex(q^2 + q + 1, C_4) = q(q + 1)^2/2$$

Furedi[15] proved that

Theorem 3.4 (Furedi[15]) Let G be a quadrilateral-free graph with e edges on $q^2 + q + 1$ vertices, and suppose that $q \geq 15$. Then $e \leq q(q + 1)^2/2$.

Corollary 3.5 (Furedi[15]) If q is a prime power greater than 13, $n = q^2 + q + 1$. Then

$$ex(n, C_4) = q(q + 1)^2/2.$$

Furedi, Naor and Verstraete[16] proved that

Theorem 3.6 (Furedi, Naor and Verstraete[16])

$$ex(n, C_6) > 0.5338n^{4/3}$$

for infinitely many n and

$$ex(n, C_6) < 0.6272n^{4/3}$$

if n is sufficiently large.

The survey article on this Erdos conjecture can be found in Chung[5].

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